PMM U.S.S.R., Vol. 46, No. 5, pp. 674-678, 1983.
Printed in Great Britain.

0021-8928/83 $7.50+0.00

UDC 539.3

VARIATIONAL FORMULATION OF A CONTACT PROBLEM FOR LINEARLY ELASTIC AND PHYSICALLY NONLINEAR SHALLOW SHELLS*

G.I. L'VOV

A variational formulation is proposed for the problem of thin shell interaction with a smooth absolutely rigid stamp without taking friction into account in the contact domain. The shell material can be linearly or nonlinearly elastic. (Elastic-plastic problem /1/ can be reduced to the latter case under certain assumptions). Application of the Lions–Stampachia method of variational inequalities reduces the task to the problem of minimizing a Lagrange functional in a set of allowable displacements. The existence and uniqueness of the solution are proved under definite assumptions about the properties of the strain diagram.

Investigation of contact problems for finite size bodies by the Lions– Stampachia method of variational inequalities was executed in /2,3/. Problems on the bending of thin plates with unilaterial constraints were examined in /4/.

1. Formulation of the problem. A shallow thin shell is considered, whose middle surface occupying a manifold S^* is considered sufficiently smooth and representable by an equation in parametric form

$$\mathbf{R} = \mathbf{R}(x_1, x_2) \in C^{(2)}(S) \tag{1.1}$$

which performs a homeomorphic mapping of the middle surface S^* into the domain S of the plane x_1x_2. The domain S is a finite sum of bounded star domains, its boundary L consists of a finite number of closed contours of Liapunov class.

The shell can be contiguous to an absolutely rigid stamp over part of a surface whose mapping on the plane x_1x_2 will be denoted by S_k. The boundary L_k of this domain and the pressure between the stamp and the shell are to be determined. The rest of the domain is denoted by S_0 so that $S = S_0 U S_k$.

We assume that there is no friction in the contact domain and the stamp surface is smooth, i.e., has a continuously rotating tangent plane in the zone where contact is possible. We set the stamp surface by an equation in the Cartesian coordinates of its points

$$f(x_1, x_2, x_3) = 0 \tag{1.2}$$

For definiteness, we assume that within the stamp $f < 0$ while $f > 0$ outside.

For thin shells it can be assumed that the equation of the shell surface with which stamp contact is possible will have the form (1.1). Then the condition

$$f(\mathbf{R} + \mathbf{U}) \geqslant 0 \tag{1.3}$$

should be satisfied after shell deformation, where $\mathbf{U} = \mathbf{U}(x_1, x_2)$ is the displacement vector of points of the middle surface. Considering the displacements small and assuming that $|\operatorname{grad} f| > 0, \ \forall (x_1, x_2) \in S_k$, we linearize condition (1.3) with respect to U

$$f(\mathbf{R}) + \mathbf{U} \operatorname{grad} f(\mathbf{R}) \geqslant 0, \ \forall (x_1, x_2) \in S \tag{1.4}$$

We set the boundary conditions on L for a clamped edge

$$u_1 = u_2 = w = 0, \ w_{,i} n_i = 0 \tag{1.5}$$

Here u_1, u_2, w are components of the vector $\mathbf{U}$ in a local basis of the curvilinear coordinate system x_1, x_2 the subscript after the comma denotes differentiation with respect to the appropriate coordinate, and n_i are components of the external normal to the contour L.

Conditions (1.4) and (1.5) impose additional constraints on the displacements $\mathbf{U}$, taken as kinematically allowable.

---

*Prikl.Matem.Mekhan.,46,No.5,841-846,1982

To formulate the contact problem, we use the fundamental relationships of the linear theory of shallow shells.

$$\varepsilon_{ij} = {}^1/_2(u_{i,j} + u_{j,i}) + k_{ij}w, \quad \varkappa_{ij} = -w_{,ij}$$
$$e_{ij} = \varepsilon_{ij} + z\varkappa_{ij}, \quad T_{ij} = A_{ijkl}\varepsilon_{kl}, \quad M_{ij} = D_{ijkl}\varkappa_{kl} \tag{1.6}$$
$$A_{ijkl} = hc_{ijkl}, \quad D_{ijkl} = (h^3/12)c_{ijkl} \quad (i, j, k, l = 1, 2)$$

Here ε_{ij} and $\varkappa_{ij}$ are components of the tangential and bending strain tensors of the middle surface, e_{ij} is the stain tensor of an arbitrary point of the shell, T_{ij}, M_{ij} are the force and moment tensors, h is the shell thickness, and c_{ijkl} is a symmetric, positive-definite tensor of the elastic constants of the materials. For an isotropic shell material

$$c_{ijkl} = \frac{E}{1-\mu^2}\left[\frac{1-\mu}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \mu\delta_{ij}\delta_{kl}\right]$$

We write the equilibrium equations in the form

$$T_{ij,j} + p_i = 0, \quad (x_1, x_2) \in S$$
$$M_{ij,ij} - K_{ij}T_{ij} + P_3 = 0, \quad (x_1, x_2) \in S_0 \tag{1.7}$$
$$M_{ij,ij} - K_{ij}T_{ij} + P_3 + q = 0, \quad (x_1, x_2) \in S_k$$

Here p_i are the projections of the external load in the local basis, and $q \geqslant 0$ is the contact pressure between the shell and the stamp.

The initial problem is to determine the vector-function $\mathbf{U} = \mathbf{U}(x_1, x_2)$ that satisfies the complete system of shallow shell theory equations (1.6) and (1.7), the boundary conditions (1.5), and the constraint (1.4). The set of points at which (1.4) is satisfied with the equality sign forms the desired contact zone.

We limit ourselves to cases in which the boundary L_k of the contact domain is a closed curve without angular points, and the characteristic dimensions of the contact domain exceed the shell thickness significantly.

2. Reduction of the problem to a variational problem. Let us introduce the S.L. Sobolev space W for the vector functions $\mathbf{U}\{u_1, u_2, w\}$

$$W = W_2^1(S) \times W_2^1(S) \times W_2^2(S) \tag{2.1}$$

We denote the space of functions from W that satisfy conditions (1.5) on L by W°. As is shown in /5/, the equivalent norm in the space W° is generated by the scalar product

$$(\mathbf{U}, \mathbf{v}) = \int_S [A_{ijkl}\varepsilon_{ij}(\mathbf{U})\varepsilon_{kl}(\mathbf{v}) + D_{ijkl}\varkappa_{ij}(\mathbf{U})\varkappa_{kl}(\mathbf{v})]\, dS$$

We define a closed, convex set K a symmetric bilinear $\pi(\mathbf{U}, \mathbf{v})$, and a linear $L(\mathbf{U})$ functional in the space W°

$$K = \{\mathbf{v}/\mathbf{v} \in W^\circ;\ f(\mathbf{R}) + \mathbf{v}\ \mathrm{grad}\, f(\mathbf{R}) \geqslant 0, \quad \forall\, (x_1, x_2) \in S\}$$
$$\pi(\mathbf{U}, \mathbf{v}) = \int_S [A_{ijkl}\varepsilon_{ij}(\mathbf{U})\varepsilon_{kl}(\mathbf{v}) + D_{ijkl}\varkappa_{ij}(\mathbf{U})\varkappa_{kl}(\mathbf{v})]\, dS, \tag{2.2}$$
$$L(\mathbf{U}) = \int_S \mathbf{PU}\, dS$$

where $\mathbf{P}\{p_1, p_2, p_3\}$ is the vector field of the external forces acting on the shell middle surface.

Theorem 1. The vector-function $\mathbf{U} \in K$ will be a solution of the initial contact problem if and only if the following variational inequality is satisfied

$$\pi(\mathbf{U}, \mathbf{U}^\circ - \mathbf{U}) \geqslant L(\mathbf{U}^\circ - \mathbf{U}), \quad \forall \mathbf{U}^\circ \in K, \quad \mathbf{U} \in K \tag{2.3}$$

Proof. Let $\mathbf{U}$ be the solution of the initial problem. We multiply (1.7) scalarly by $\mathbf{U}^\circ - \mathbf{U}$, where $\mathbf{U}^\circ$ is any element from K, we integrate the expression obtained with respect to S, and we manipulate it by using the Green's formula. Taking (1.6) into account, we obtain

$$\pi(\mathbf{U}, \mathbf{U}^\circ - \mathbf{U}) = L(\mathbf{U}^\circ - \mathbf{U}) + \int_S q(w^\circ - w)\, dS \tag{2.4}$$

The integral in (2.4) is taken substantially over S_k since $q = 0$ for $(x_1, x_2) \in S_0$. For the domain S_k

$$f(\mathbf{R}) + \mathbf{U} \operatorname{grad} f(R) = 0 \tag{2.5}$$

An arbitrary element $U^\circ \in K$ satisfies the inequality

$$f(\mathbf{R}) + \mathbf{U}^\circ \operatorname{grad} f(R) \geqslant 0, \ \forall (x_1, x_2) \in S_k \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$(\mathbf{U}^\circ - \mathbf{U}) \operatorname{grad} f(\mathbf{R}) \geqslant 0, \ \forall (x_1, x_2) \in S_k \tag{2.7}$$

In the contact zone $\operatorname{grad} f(\mathbf{R}) = \mathbf{n} |\operatorname{grad} f(\mathbf{R})|$, where $\mathbf{n}$ is the unit normal vector of the shell middle surface in the deformed state.

No difference is made in geometrically linear shell theory between the local bases of the initial and deformed middle surface, hence

$$(\mathbf{U}^\circ - \mathbf{U}) \operatorname{grad} f(\mathbf{R}) = (\mathbf{U}^\circ - \mathbf{U}) \mathbf{n} \mid \operatorname{grad} f(R) \mid = (w^\circ - w) \mid \operatorname{grad} f(\mathbf{R}) \mid$$

Since $|\operatorname{grad} f(\mathbf{R})| > 0$, then $(w^\circ - w) \geqslant 0$ follows from (2.7). Therefore, the integral in (2.4) is non-negative $(q \geqslant 0)$, and the element $\mathbf{U}$ satisfies the variational inequality (2.3).

Now, let $\mathbf{U}$ be the solution of the variational inequality (2.3). We introduce the set $D(S)$ of infinitely differentiable functions $\boldsymbol{\varphi}\{\varphi_1, \varphi_2, \varphi_3\}$ with medium compact in S. The set $D(S)$ is compact in W°.

Let us first examine the case when $\boldsymbol{\varphi}$ takes on values from the set $\Phi = \{\boldsymbol{\varphi} \mid \boldsymbol{\varphi} \in D(S), \ \varphi_3 = 0\}$. The functions $\mathbf{U} \pm \varepsilon\boldsymbol{\varphi} \in K$ for sufficiently small ε, hence, by substituting $\mathbf{U}^\circ = \mathbf{U} + \varepsilon\boldsymbol{\varphi}$ and $\mathbf{U}^\circ = \mathbf{U} - \varepsilon\boldsymbol{\varphi}$ in (2.3), we obtain

$$\pi(\mathbf{U}, \boldsymbol{\varphi}) \geqslant L(\boldsymbol{\varphi}), \quad -\pi(\mathbf{U}, \boldsymbol{\varphi}) \geqslant -L(\boldsymbol{\varphi})$$

It hence follows that for any element $\boldsymbol{\varphi}$ the equality $\pi(\mathbf{U}, \boldsymbol{\varphi}) = L(\boldsymbol{\varphi})$ is satisfied, which is reduced by the use of (1.6) and the Green's formula to

$$\int_L T_{ij}\varphi_i n_j \, dl - \int_S T_{ij,j}\varphi_i \, dS = \int_S p_i \varphi_i \, dS \quad (i, j = 1, 2) \tag{2.8}$$

Since $\varphi_i = 0$, $\forall (x_1, x_2) \in L$, then it follows from (2.8) that u_1, u_2 is the generalized solution of the first equilibrium equation (1.7).

Now, let $\boldsymbol{\varphi}$ take on values from the set

$$\Phi_0 = \{\boldsymbol{\varphi} \mid \boldsymbol{\varphi} \in D(S_0), \quad \varphi_{3,i} = 0, \quad \forall (x_1, x_2) \in L, \ L_k, \\ \varphi_1 = \varphi_2 = 0\}$$

Substituting the arbitrary element $\mathbf{U}^\circ = \mathbf{U} \pm \varepsilon\boldsymbol{\varphi}$ into (2.3), we obtain $\pi(\mathbf{U}, \boldsymbol{\varphi}) = L(\boldsymbol{\varphi})$. Taking account of the properties of $\boldsymbol{\varphi}$ and applying the Green's formula twice, we have

$$\int_{S_0} (T_{ij}K_{ij} - M_{ij,ij} - P_3)\varphi_3 \, dS = 0$$

It hence follows that $\mathbf{U}$ is a generalized solution of the second equilibrium equation (1.7).

Finally, let us examine the case when φ takes on values from

$$\Phi_k = \{\boldsymbol{\varphi} \mid \boldsymbol{\varphi} \in D(S_k), \ \varphi_3 \geqslant 0, \ \varphi_{3,i} = 0, \ \forall (x_1 x_2) \in L_k, \ \varphi_1 = \varphi_2 = 0\}$$

For sufficiently small $\varepsilon > 0$ the element $\mathbf{U}^\circ = \mathbf{U} + \varepsilon\boldsymbol{\varphi} \in K$. Substituting this element into (2.3), after analogous manipulations to those before, we obtain that the contact pressure q between the shell and the stamp is non-negative. The magnitude of this pressure can be found from the third equation in (1.7).

To give a foundation to the manipulations performed, it must be assumed that

$$p_i \in L_2, \quad M_{ij,ij} \in L_2(S_k), \quad M_{ij,ij} \in L_2(S_0), \quad T_{ij,j} \in L_2(S) \tag{2.9}$$

The solution of the variational inequality (2.3) is equivalent to the problem minimizing the quadratic functional /6/

$$J(\mathbf{v}) = {}^1/_2\pi(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}), \quad \mathbf{v} \in K \tag{2.10}$$

It is known /6/ that a unique element $\mathbf{U} \in K$, the solution of the problem

$$J(\mathbf{U}) = \inf_{\mathbf{v} \in K} J(\mathbf{v})$$

exists for a continous positive-definite symmetric functional $\pi(v, v)$ in W°. This element is also characterized by the variational inequality (2.3).

3. Contact problem for a physically nonlinear shell. Let the relation between the stress and strain for an incompressible shell material be given in the form /1/

$$S_{ij} = {}^2/_3\Phi(\varepsilon_0)\, e_{ij}, \quad \Phi(\varepsilon_0) = \frac{\sigma_0}{\varepsilon_0} \quad (i, j = 1, 2, 3) \tag{3.1}$$
$$S_{ij} = \sigma_{ij} - \delta_{ij}\sigma, \quad \sigma = {}^1/_3\delta_{ij}\sigma_{ij}, \quad \sigma_0 = ({}^3/_2 S_{ij}S_{ij})^{1/2}, \quad \varepsilon_0 = ({}^2/_3 e_{ij}e_{ij})^{1/2}$$

Here S_{ij} is the stress deviator, e_{ij} is the strain tensor components, σ is the mean stress, σ_0, ε_0 are the stress and strain intensities.

For a physically nonlinear shell the initial problem (1.4)—(1.7) will include the following relationships for the force and moments (the integrals are taken between the limits $-h/2$ and $h/2$) /1/:

$$T_{ij} = \int \sigma_{ij}\, dz = (J_1\varepsilon_{kl} + J_2\varkappa_{kl})\, b_{ijkl} \tag{3.2}$$
$$M_{ij} = \int \sigma_{ij} z\, dz = (J_2\varepsilon_{kl} + J_3\varkappa_{kl})\, b_{ijkl}$$
$$J_n = {}^2/_3 \int \Phi(\varepsilon_0)\, z^{n-1}\, dz, \quad b_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl}$$

Let us introduce a nonlinear functional in W°

$$J(\mathbf{U}) = \int_V \int_0^{\varepsilon_0(\mathbf{U})} \sigma_0(t)\, dt\, dV - \int_S \mathbf{PU}\, dS \tag{3.3}$$

in which the first integral is taken over the whole volume of the shell.

The functional derivative (Gateaux) with respect to the direction v is

$$J'(U, \mathbf{v}) = \int_V \sigma_{ij}(\mathbf{U})\, e_{ij}(\mathbf{v})\, dV - \int_S \mathbf{Pv}\, dS \quad (i, j = 1, 2) \tag{3.4}$$

Theorem 2. The solution of the contact problem in a physically nonlinear formulation is equivalent to the solution of the variational inequality

$$J'(\mathbf{U}, \mathbf{U}^\circ - \mathbf{U}) \geqslant 0, \quad \forall \mathbf{U}^\circ \in K, \quad U \in K \tag{3.5}$$

Proof. Let $\mathbf{U}$ be the solution of the initial problem. We multiply (1.7) scalarly by $(\mathbf{U}^\circ - \mathbf{U})$, integrate the result with respect to S, and manipulate it by using the Green's formula with (1.6) and (3.2) taken into account. We obtain

$$\int_V \sigma_{ij}(\mathbf{U})\, e_{ij}(\mathbf{U}^\circ - \mathbf{U})\, dV - \int_S \mathbf{P}(\mathbf{U}^\circ - \mathbf{U})\, dS = \int_S q(w^\circ - w)\, dS$$

Taking account of the non-negativity of the last integral (see Sect.2), we obtain that $J'(\mathbf{U}, \mathbf{U}^\circ - \mathbf{U}) \geqslant 0$.

Now, let $\mathbf{U}$ be the solution of the variational equation (3.5). We introduce an arbitrary element $\varphi \in \Phi$. For sufficiently small $\varepsilon > 0$ we have $\mathbf{U} \pm \varepsilon\varphi \in K$. The functional $J'(\mathbf{U}, \mathbf{v})$ is linear in $\mathbf{v}$, hence, by substituting $\mathbf{U}^\circ = \mathbf{U} + \varepsilon\varphi$, $\mathbf{U}^\circ = \mathbf{U} - \varepsilon\varphi$ in (3.5), we obtain $J'(\mathbf{U}, \varphi) = 0$ or

$$\int_V \sigma_{ij}(\mathbf{U})\, e_{ij}(\varphi)\, dV - \int_S \mathbf{P}\varphi\, dS = 0$$

From here taking (3.2) into account, by using the Green's formula we hence obtain

$$\int_S [T_{ij,j}(\mathbf{U})\, \varphi_i + p_i\varphi_i]\, dS = 0$$

Therefore, $T_{ij}(\mathbf{U})$ is the generalized solution of the first equilibrium equation in (1.7).

Now, let $\varphi \in \Phi_0$. Substituting $\mathbf{U}^\circ = \mathbf{U} \pm \varepsilon\varphi$ into the inequality (3.5), we again obtain $J'(\mathbf{U}, \varphi) = 0$.

Taking account of the properties of φ and the relationship (3.2), then applying the Green's formula twice, we analogously obtain that $\mathbf{U}$ is the solution of the second equilibrium equation in (1.7).

And finally we take $\varphi \in \Phi_k$. The element $\mathbf{U}^\circ = \mathbf{U} + \varepsilon\varphi \in K$ for sufficiently small $\varepsilon > 0$. Subsituting such an element into (3.5), applying the Green's formula, and taking account of the third equilibrium equation in (1.7), we obtain that

$$q \geqslant 0, \quad \forall (x_1, x_2) \in S_k$$

The manipulations performed are based on the assumptions (2.9). The solution of the variational inequality (3.5) is equivalent to the problem of minimizing the functional

$$J(\mathbf{U}) \leqslant J(\mathbf{v}), \quad \forall \mathbf{U} \in K \tag{3.6}$$

The proof of this assertion reduces to confirming the convexity and differentiability of the functional $J(\mathbf{v})$ /7/. Strict convexity of the functional (3.3) is proved analogously /2/.

To prove the existence and uniqueness of the solution of the problem (3.6) of minimization of a strictly convex functional, it is sufficient to prove the coercivity of the functional (3.3), i.e., that

$$\lim J(\mathbf{v}) = +\infty, \|\mathbf{v}\| \to +\infty$$

The norm in the space W°, generated by the scalar product (2.2), can be represented for isotropic incompressible material in the form (G is the shear modulus of the material)

$$\|\mathbf{v}\|^2 = (\mathbf{v}, \mathbf{v}) = \int_V C_{ijkl} e_{ij} e_{kl} dV = 3G \int_V \varepsilon_0^2 dV$$

Under the assumption that $\sigma_0(\varepsilon_0)$ is a strictly concave monotonically increasing function, and $\sigma_0(\varepsilon_0) \leqslant 3G\varepsilon_0$, $\forall \varepsilon_0$, analogously to /2/, the coercivity of the corresponding functional is proved for the Hertz problem.

The approach elucidated above is convenient in cases when the stamp displacement is known. If the external force transmitted through a rigid stamp to a shell or plate is known /8,9/, then the work of these forces on the required displacement of the stamp must be taken into account for a variational formulation in functionals of the type (2.10) and (3.3).

REFERENCES

1. IL'IUSHIN A.A., Plasticity. Gosteoretizdat, Moscow – Leningrad, 1948.
2. KRAVCHUK A.S., On the Hertz problem for linearly and nonlinearly elastic bodies of finite dimensions, PMM, Vol.41, No.2, 1977.
3. KRAVCHUK A.S., Formulation of a problem of contact beteen several deformable bodies as a nonlinear programming problem, PMM, Vol.42, No.3, 1978.
4. DUVAUT G. and LIONS J-O., Les inéquations en mécanique et en physique, 1972.
5. VOROVICH I.I., LEBEDEV L.P. and SHLAFMAN Sh.M., On some direct methods and the existence of a solution in the nonlinear theory of elastic nonshallow shells of revolution, PMM, Vol.38, No.2, 1974.
6. LIONS J.-L., Certain Methods of Solving Nonlinear Boundary Value Problems. MIR, Moscow, 1972.
7. CEAT G., Optimization. Theory and Algorithms. MIR, Moscow, 1973.
8. GALIN L.A., Contact Problems of Elasticity and Viscoelasticity Theories, NAUKA, Moscow, 1980.
9. GRIGOLIUK E.I. and TOLKACHEV V.M., Contact Problems of the Theory of Plates and Shells. Mashinostroenie, Moscow, 1980.

Translated by M.D.F.